On Local Constraints of D=4 Euclidean Supergravity in Terms of Dirac Eigenvalues

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Abstract

It has been recently shown that in order to have Dirac eigenvalues as observables of Euclidean supergravity, certain constraints should be imposed on the covariant phase space as well as on Dirac eigenspinors. We investigate the relationships among the constraints in the first set and argue that these relationships are not linear. We also derive a set of equations that should be satisfied by some arbitrary functions that enter as coefficients in the equation expressing the linear dependency of the constraints in order that the second set of constraints be linearly independent.

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It has been recently shown that Dirac eigenvalues can be used as observables of D=4 Euclidean gravity on compact spacetime without boundary[1,2]. This result is based on previous works done in the framework of the noncommutative geometry as an underlying structure of gravity beyond the Planck scale[3]. To extend this idea to supergravity, it is necessary to impose some constraints, called primaries, on the set of Dirac eigenspinors[5]. These constraints allow us to interpret Dirac eigenvalues as local observables of N=1 D=4 Euclidean supergravity. To promote them to global observables, one has to impose the compatibility of the geometrical structure of the spacetime with the two sets of constraints. That results in severe restrictions on spacetime manifolds that admit this type of global observables[5]. However, we should note that this discussion is related to the problem of realizing the Dirac operator in curved space. At present, no satisfactory answer to this problem is known, even though there are many studies on the dirac operator that rely on different coordinates in spacetime. It is not our purpose to solve this problem here which is clearly a nontrivial one.

The aim of this letter is to discuss the relationships among the primaries as well as the secondaries. The basic motivation for this relies on the fact that all of the previous analysis of this system is done at the classical level, while the main reason for introducing this kind of description of Euclidean supergravity aims at giving insights in the quantum theory.This amounts to applying one of the quantization methods based on the BRST symmetry, like BV-BRST or BFV-BRST, which are the most powerful methods for quantizing the constrained systems[6]. However, as was noticed previously and as will result from the present discussion, this is a difficult task. Therefore, in this paper we limit our analysis to the first essential step of the quantization of the system, namely to the discussion of the nontrivial relationships among the constraints.

Let us consider N=1 D=4 Euclidean supergravity on a compact (spin) manifold without boundary. The on-shell supergraviton is given by the vierbein fields $e_\mu^a(x)$, where $\mu = 1, 2, 3, 4$ and the gravitino fields ψ_μ^α where α is an index for an $SO(4)$ spinor that satisfies $\bar{\psi} = \psi^T C$[7]. This is our 'Majorana' spinor in the case of Euclidean supergravity used because $SO(4)$ does not admit an usual Majorana representation. We can also work with symplectic spinors[8]. The gauge group is given by four dimensional diffeomorphisms, local $SO(4)$ rotations and N=1 local supersymmetry and its action on the supergraviton is given by

$$\delta e_\mu^a = \xi^\nu \partial_\nu e_\mu^a + \theta^{ab} e_{b\mu} + \frac{1}{2}\bar{\epsilon}\gamma^a \psi_\mu \tag{1}$$

$$\delta \psi_\mu^\alpha = \xi^\nu \partial_\nu \psi_\mu^\alpha + \theta^{ab}(\sigma_{ab})^\alpha_\beta \psi_\mu^\beta + \mathcal{D}_\mu \epsilon^\alpha \tag{2}$$

where $\xi = \xi^\nu \partial_\nu$ is an infinitesimal vector field on M, $\theta_{ab} = -\theta_{ba}$ parametrize an infinitesimal rotation and ϵ is an infinitesimal Majorana spinor field. Here $\mathcal{D}_\mu$ is the nonminimal covariant derivative acting on spinors, associated to the usual minimal one. The covariant phase space is defined to be the space of the solutions of the equations of motion modulo the gauge transformations. Then the observables of the theory are functions on the phase space.

In the presence of local supersymmetry, the Dirac operator is given by

$$D = i\gamma^a e_a^\mu (\partial_\mu + \frac{1}{2}\omega_{\mu bc}(e,\psi)\sigma^{bc}), \tag{3}$$

where $\sigma^{bc} = \frac{1}{4}[\gamma^a, \gamma^b]$, γ^a's form a representation of Clifford algebra $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ and $\omega_{\mu bc}(e,\psi)$ are the components of spin connection in the presence of local supersymmetry.

The Dirac operator is a first order elliptic operator on M. Thus, since M is compact, D has a discrete spectrum

$$D\chi^n = \lambda^n \chi^n, \tag{4}$$

where $n = 0, 1, 2, \ldots$. The eigenvalues λ^n's are functions on the space of all supermultiplets (e, ψ). In order to be interpreted as observables, they must also be gauge invariant. The invariance of λ^n's under diffeomorphisms, local $SO(4)$ rotations and local supersymmetry, is expressed by the following set of equations[4]

$$\mathcal{T}^{n\mu}_{a} \partial_\nu e^a_\mu - \Gamma^{n\mu}_{\alpha} \partial_\nu \psi^\alpha_\mu = 0, \tag{5}$$

$$\mathcal{T}^{n\mu}_{a} e_{b\mu} + \Gamma^{n\mu} \sigma_{ab} \psi_\mu = 0, \tag{6}$$

$$\mathcal{T}^{n\mu}_{a} \bar{\epsilon} \gamma^a \psi_\mu + \Gamma^{n\mu} \mathcal{D}_\mu \epsilon = 0, \tag{7}$$

where

$$\mathcal{T}^n_\mu = \langle \chi^n | (\frac{\delta}{\delta e^a_\mu} D) | \chi^n \rangle \quad , \quad \Gamma^{n\mu}_{\alpha} = \langle \chi^n | (\frac{\delta}{\delta \psi^\alpha_\mu} D) | \chi^n \rangle, \tag{8}$$

are the functional derivatives of the Dirac operator with respect to the graviton and gravitino, respectively. The scalar product is naturally defined in the spinor bundle on M

$$< \psi, \chi > = \int \sqrt{g} \psi^* \chi. \tag{9}$$

Consistency requires that the following constraints be imposed on the set of Dirac eigenspinors[4]

$$\{[b^\mu(\xi) - c(\lambda\xi)^\mu] \partial_\mu + f(\xi)\} \chi^n = 0, \tag{10}$$

$$[\theta^a_a D - g(\theta) + h(\theta)] \chi^n = 0. \tag{11}$$

$$[j^\mu_a(\epsilon) \partial_\mu + k_a(\epsilon) + l_a] \chi^n = 0 \tag{12}$$

where the following shorthand notations have been employed

$$\begin{aligned}
b^\mu(\xi) = i\gamma^a b^\mu_a(\xi) \quad &, \quad b^\mu_a(\xi) = \xi^\nu \partial_\nu e^\mu_a - e^\nu_a \partial_\nu \xi^\mu - 2 e^\nu_a \xi^\mu \omega_{\nu bc} \sigma^{bc} \\
c(\lambda, \xi)^\mu = (\lambda^n - D) \xi^\mu \quad &, \quad f(\xi) = i\gamma^a \xi^\nu \partial_\nu (e^\mu_a \omega_{\mu bc}) \sigma^{bc} \\
c(\lambda, \xi)^\mu = (\lambda^n - D) \xi^\mu \quad &, \quad g(\theta) = [\gamma^c e^\mu_c ([\theta\sigma, \omega_{\mu ab}] - \partial_\mu \theta \sigma M_{ab})] \sigma^{ab} \\
h(\theta) = i(\lambda^n - D)\theta\sigma \quad &, \quad j^\mu_a(\epsilon) = \frac{1}{2} \gamma_a \bar{\epsilon} \psi^\mu \quad , \quad k_a(\epsilon) = \frac{1}{2} \gamma_a \bar{\epsilon} \psi^\mu \omega_{\mu cd} \sigma^{cd} \\
l_a = e^\mu_a [A_{\mu cd} - \frac{1}{2} e_{\mu d} A^e_{ec} + \frac{1}{2} e_{\mu c} A^e_{ed}] \sigma^{cd} \quad &, \quad A^{\mu\nu}_a = \bar{\epsilon} \gamma_5 \gamma_a \mathcal{D}_\lambda \psi_\rho \epsilon^{\nu\mu\lambda\rho}.
\end{aligned} \tag{13}$$

The first set of constraints, (5), (6) and (7), also called primaries, should be imposed on the supergravitons. Equations (10), (11) and (12) follow as a consequence of the primaries, and therefore are called secondaries. As was note in[5], primaries as well as secondaries should be taken into account when the BRST quantization of the system is performed. For example, both of the sets of constraints determine the partition function in the path integral approach. Therefore, it is crucial to elucidate the reducibility of this system. To this end we have to address the question of linear dependency of the two sets of constraints.

Let us begin by discussing the relationships among the primaries. To simplify the expressions in what follows, let us denote the constraints (5), (6) and (7) by $\Sigma^n_\nu(\mathcal{T},\Gamma) = 0$, $\Theta^n_{ab}(\mathcal{T},\Gamma) = 0$ and $\Phi^n(\mathcal{T},\Gamma) = 0$, respectively. Now, if we multiply Θ^n_{ab} at left by $\bar{\epsilon}\gamma^a$ and sum over a, and then multiply the result at right by $e^b_\rho\psi^\rho$, we come across the first term in Φ^n. Thus we can immediately write down a relationship between the second and the third primary

$$\Phi^n - \bar{\epsilon}\gamma^a\Theta^n_{ab}\psi^b = \Gamma^{n\mu}\mathcal{D}_\mu\epsilon - \bar{\epsilon}\gamma^a\Gamma^{n\nu}\sigma_{ab}\psi_\mu\psi^b \tag{14}$$

Eq. (5) does not express, in general, the linear independency of Θ^n_{ab} and Φ^n. This is true only if the arbitrary spinor ϵ obeys the following equation

$$\mathcal{D}_\mu\epsilon = \bar{\epsilon}\gamma^a\Gamma^{n\nu}\sigma_{ab}\psi_\mu\psi^b, \tag{15}$$

which selects from general local supersymmetry transformations a certain class of transformations of which parameters obey (15). We can also see that there is no other linear relationship among these constraints based on the second terms in Θ^n_{ab} and Φ^n and therefore we conclude that these two constraints are not linearly independent.

In a similar manner we can approach the relationship between Σ^n_ν and Θ^n_{ab}. In this case, we see that Σ^n_ν involves the derivatives of the supergraviton, while Θ^n_{ab} involves only its components, besides, of course, the $\mathcal{T}^{n\mu}_a$ and $\Gamma^{n\mu}_\alpha$ terms which both involve integrals of supergraviton and its derivatives. Therefore, we cannot obtain a linear relationship between these two constraints. However, we can obtain, after some simple algebra, the following equation

$$\Sigma^n_\nu - \partial_\nu\Theta^n_{ab} = \partial_\nu(\Gamma^{n\mu}_\alpha(\sigma^a_a)^\alpha_\beta\psi^\beta_\mu) - \Gamma^{n\mu}_\alpha\partial_\nu\psi^\alpha_\mu - \partial_\nu\mathcal{T}^{n\mu}_a e^a_\mu. \tag{16}$$

In the right-hand side of Eq.(16) we have written down explicitely the spinor index in the first term. This shows us that another relationship based on the identification of the last two terms, modulo some functions, is not possible.

Because the relationship between Φ^n and Θ^n_{ab} implies the derivation of the later and because (14) expresses the relationship between Φ^n and Θ^n_{ab}, we see that a relationship between Φ^n and Σ^n_ν would imply the derivatives of Φ^n. A final relationship among all the constraints and thus between Φ^n and Σ^n_ν too, is also nonlinear and is given by

$$\begin{aligned}\partial_\nu\bar{\epsilon}\gamma^a\Theta^n_{ab}\psi^b + \bar{\epsilon}\gamma^a\Theta^n_{ab}\partial_\nu\psi^b + \sum_{a\neq b}\bar{\epsilon}\gamma^a\partial_\nu\Theta^n_{ab}\psi^b - \partial_\nu\Phi^n + \sum_a\bar{\epsilon}\gamma^a\sigma^n_\nu\psi_a = \\ \sum_a(\bar{\epsilon}\gamma^a\partial_\nu(\Gamma^{n\nu}\sigma_{aa}\psi_\mu) + \Gamma^{n\mu}\partial_\nu\psi_\mu + \partial_\nu\mathcal{T}^{n\mu}_a e_{a\mu})\psi^a - -\partial_\nu(\Gamma^{n\mu}\mathcal{D}_\mu\epsilon - \bar{\epsilon}\gamma^a\Gamma^{n\nu}\sigma_{ab}\psi_\nu\psi^b).\end{aligned} \tag{17}$$

Some comments are in order now. As we can see from (14), (16) and (17), the set of primaries is not linearly independent. However, as we have already noticed, for a particular

set of local supersymmetries given by the solutions of (15), (14) turn into a linear equation among the three primaries, where the third one appears with a null coefficient. The functions multiplying Φ^n and Θ^n_{ab} are the ones in the left-hand side of (5). This has some important consequences in the BRST quantization. Indeed, in the partition function of Fadeev-Popov quantization method, the constraints $\Sigma^n_\nu(\mathcal{T},\Gamma)=0$, $\Theta^n_{ab}(\mathcal{T},\Gamma)=0$ and $\Phi^n(\mathcal{T},\Gamma)=0$ enter the exponential of the action through a term of the following form

$$S_c=\int(\sigma^\nu_n\Sigma^n_\nu+\tau^{ab}_n\Theta^n_{ab}+\phi_n\Phi^n), \tag{18}$$

where σ^ν_n, τ^{ab}_n and ϕ_n associated to the gauge averaging conditions. The corresponding ghosts, denoted by s^ν_v, t^{ab}_n and f_n enter the exponential through the following action

$$S_{gh}=\int(s^\nu_v\delta_\alpha\Sigma^n_\nu+t^{ab}_n\delta_\alpha\theta^n_{ab}+f_n\delta_\alpha\phi^n)c^\alpha, \tag{19}$$

where c^α are the ghosts associated to the gauge transformations, generally denoted by δ_α. Now, if for certain supersymmetry transformations, i. e. those for which the parameters satisfy (15), the constraints become linearly independent, the theory, irreducible up to now, becomes a first reducible theory. Thus we obtain an enhancement of the content of ghost-antighost fields. If we denote the linear constraint by $\mathcal{C}(\Sigma^n_\nu,\Theta^n_{ab},\Phi^n)=0$ we have the corresponding ghost-antighost structure associated to this equation[6].

Another issue is what the relationships among the secondaries are. The secondaries restrict the set of eigenspinors and implicitely the possible $\mathcal{T}^{n\mu}_a$'s and $\Gamma^{n\mu}_\alpha$'s, respectively, which are diagonal matrix elements on eigenspinors. Therefore, taking into account the secondaries, the number of Σ^n_ν's, Θ^n_{ab}'s and Φ^n's in (18) and (19) should reduce. Thus we can see that investigating the dependence of secondaries is important even if these constraints are directly imposed on the covariant phase space.

Let us denote (10), (11) and (12) by $\mathcal{C}_1\chi^n=0$, $\mathcal{C}_2\chi^n=0$ and $\mathcal{C}_3\chi^n=0$, respectively. We are looking for a linear relationship among all of the secondaries. Now if we consider $\mathcal{C}_i$'s as some algebraic function of even Grassman parity, we obtain after some simple algebra the following equations

$$e^\rho_c j^\nu_d(b^\mu+c^\mu)f_{1\rho}f^d_{1\nu}+(b^\nu+c^\nu)j^\rho_d e^\mu_c f_{2\nu}f^d_{2\rho}+(b^\nu+c^\nu)e^\rho_c j^\mu_d f_{3\nu}f_{3\rho}f^d_3=0 \tag{20}$$

and

$$\begin{aligned}i\gamma^c\theta^a_a(e^\rho_c j^\nu_d f_{1\rho}f_{1\nu}f+(b^\nu+c^\nu)j^\rho_d\theta^a_a\gamma^c e^\mu_c\omega_{\mu ef}\sigma^{ef}f_{2\nu}f^d_{2\rho}+\\ (b^\nu+c^\nu)e^\rho_c(k_d+l_d)f_{3\nu}f_{3\rho}f^d_3)+(h-g)f_{2\nu}f^d_{2\rho}=0\end{aligned} \tag{21}$$

where $f_{1\rho}$,$f^d_{1\nu}$,$f_{2\nu}$,$f^d_{2\rho}$,$f_{3\nu}$,$f_{3\rho}$ and f^d_3 are the coefficients of the constraints in the equation that describes their linear dependency. These coefficients are in number of 32 of them and should obey the above equations. Because the number of functions exceeds that of equations, the system is not well determined. However, some particular solutions could be find in principle by fixing 30 of the functions, let say to some constants. Eqs. (20) and (21) depend on the gauge transformations. They also include some spinorial objects and thus are not trivial. If they admit nontrivial solutions then there is a linear relationship among all of the secondaries.

Now let us take two of the secondaries. A linear combination of $\mathcal{C}_1$ and $\mathcal{C}_2$ implies that the following equations hold

$$e^{\nu}_{c} g_{1\nu}(b^{\mu} + c^{\mu}) + (b^{\nu} + c^{\nu}) g_{2\nu} e^{\mu}_{c} = 0 \tag{22}$$

and

$$i\gamma^{c}\theta^{a}_{a}(e^{\nu}_{c} g_{1\nu} f + (b^{\nu} + c^{\nu}) g_{2\nu} e^{\mu}_{c} \omega_{\mu de}\sigma^{de}) - (b^{\nu} + c^{\nu}) g_{2\nu}(g - h) = 0. \tag{23}$$

Eqs. (22) and (23) must be satisfied by the two unknown spacetime vectorfields $g_{1\nu}$ and $\gamma_{2\nu}$,respectively, which are now the coefficients in the linear equation among constraints. Similarly, if we take $\mathcal{C}_2$ and $\mathcal{C}_3$, another set of two equations must hold, namely

$$j^{\nu}_{a} g_{3\nu} e^{\mu}_{c} + e^{\nu}_{c} g^{a}_{4\nu} j^{\mu}_{a} = 0 \tag{24}$$

and

$$j^{\nu}_{a} g_{3\nu}(i\theta^{a}_{a}\gamma^{c} e^{\mu}_{c}\omega_{\mu de}\sigma^{de} - g + h) + i\theta^{b}_{b}\gamma^{c} e^{\nu}_{c} g^{a}_{4\nu}(k_{a} + l_{a}) = 0, \tag{25}$$

where the unknown spacetime vectorfield is $g_{3\nu}$, while $g^{a}_{4\nu}$ is a double index unknown quantity. The final possible linear relationship holds if the following equations hold

$$j^{\nu}_{a} g^{a}_{5\nu}(b^{\mu} + c^{\mu}) + (b^{\nu} + c^{\nu}) g^{a}_{6\nu} j^{\mu}_{a} = 0 \tag{26}$$

and

$$j^{\nu}_{a} g^{a}_{5\nu} f + (b^{\nu} + c^{\nu}) g^{a}_{6\nu}(k_{a} + l_{a}) = 0, \tag{27}$$

where $g^{a}_{5\nu}$ and $g^{a}_{6\nu}$ are two indices unknown objects.

Let us make some brief comments on the equations (22) - (27). The unknown objects that must satisfy these equations represent a set of 60 unknown coefficients. In the most general case, the equations group as shown above. Then for each set of equations we have a redundant number of variables. To find some particular solutions we can fix some of them in an arbitrary way. Another fixing procedure is to consider the whole set equations as a system and then to require that some of the objects entering one of the subsets of equations be identically with some of the objects entering another subset, e. g. $g^{a}_{4\nu} = g^{a}_{5\nu}$. This amounts to impose the existence of linear relationships among *all* of the secondaries simultaneously. If the equations (20)-(27) have nontrivial solutions, the secondaries are not linear independent. As a consequence, follows the enlarging of the set of the allowed eigenspinors and implicitely the enlarging of the set of diagonal matrices in (18) and (19), respectively. We should remark that the equations derived above are not trivial and not easy to be solved in the general case. This postpone the BRST analysis of the system to some future works. We note that since local supersymmetry and $SO(4)$ invariance in supergravity coupled to matter normally require field-depended transformations for consistency, the gauge-fixing must be carried out in accordance with variations of gravitino and graviton. Their consistency with the global symmetries of the theory must be explicitely checked.

To conclude, we have analyzed in this letter the dependency of the primaries and secondaries of the Euclidean supergravity in terms of Dirac eigenvalues. We have shown that

in general the primaries are not linearly independent and thus the theory is an irreducible one in the BRST language. However, for some local supersymmetry transformations, it is possible that the theory become first stage reducible and that the ghost-antighost structure be enlarged. We have also derived the set of equations (20)-(27) which impose the linear independency of secondaries. As a consequence of these equations, the set of admissible eigenspinors is enlarged, too, and that, at its turn, modifies the extended action that enters the partition function of the system.

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